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Displacement control crack-growth instability in an elastic-softening material

Part I *Linear elastic analysis for bend specimen configuration*

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The criterion for a cusp catastrophe type of crack-growth instability in an elastic-softening material that is subjected to displacement control loading conditions has been investigated. Attention was focused on the behaviour of a material whose softening-zone size was very small in comparison with a solid's characteristic dimensions, because this facilitates a simple linear elastic analysis. Special consideration was given to the behaviour of an edge-cracked solid that was subjected to bending deformation, and the results complement those obtained by Carpinteri, who analysed the behaviour of an elastic-softening material which had a very large softening-zone size.

1. Introduction

There are several types of material such as ceramics, concretes, cements and fibre-reinforced composites where there is a softening zone behind a propagating crack tip. Within this zone a restraining stress acts between the crack faces, and this stress is related (via the material softening law) to the relative displacement of the crack faces. The restraining stress is operative until the opening at the trailing edge of the softening zone becomes sufficient for the restraining stress to fall to zero. Upon loading a solid containing a zone-free crack, the crack tip remains stationary until the stress intensity attains a critical value, K_{IC}, when the material fractures at the crack tip. Then, with continued loading, the crack extends and the restraining (softening) zone increases in size until it becomes fully developed, i.e. until the opening at the trailing edge of the zone (the original crack tip) attains a critical value. Thereafter the crack continues to extend with a constant opening at the trailing edge of the softening zone. Particular consideration has been given [1–6] to the relation between the crack tip stress intensity, K, as measured at the leading edge of the softening zone, and the crack extension; this relation depends on a variety of factors: the geometrical configuration, loading pattern, the softening law and the magnitude of K_{IC}

Carpinteri [7] has recently focused attention on the global response of an elastic-softening solid. He investigated the behaviour of a material whose fully developed softening-zone size is very large, and analysed the model of an edge-cracked solid that is subjected to three-point bending deformation. Particular consideration was given to the criterion for there to be a cusp catastrophe type of displacement control crack-growth instability, i.e. both load and displacement decrease during crack extension. By analysing a range of geometrical configurations where the solid width, length and crack depth were scaled up proportionally, Carpinteri showed that a cusp in the load–displacement record was favoured by large dimensions, and also by a small crack depth–solid width ratio. He referred to experimental results [8] which support the theoretical predictions.

The present work also investigated the problem of displacement control crack-growth instability for the bend specimen configuration, but for the case where the softening zone was very small in relation to other characteristic dimensions of the configuration, i.e. at the opposite end of the scale of material behaviour to that considered by Carpinteri [7]. The theoretical analysis defines the condition for a cusp catastrophe type of crack-growth instability, and shows how it depends on the various geometrical parameters of the configuration: solid width, solid length and crack depth.

2. The softening-zone size

Consider, like Carpinteri [7], the situation where $K_{IC} = 0$, i.e. the fracture toughness of the matrix material is presumed to be negligible. For a semi-infinite crack in an infinite solid subjected to remote loading conditions, and with Mode I plane strain deformation, it has been shown [9] that the size, R_c, of the softening zone when it is fully developed, i.e. when the opening at the trailing edge attains a critical value, δ_c, at which the restraining stress between the crack faces becomes zero, is not particularly sensitive to the details of the softening law, though it is dependent on the maximum stress, p_c, i.e. the stress at the crack tip, and δ_c. R_c is approximately equal to the value for the Dugdale–Bilby–Cottrell–Swinden (DBCS) model [10, 11] where the stress is constant within the softening zone, i.e.

$$R_c \sim \frac{0.40\, E_0 \delta_c}{p_c} \tag{1}$$

where $E_0 = E/(1 - \nu^2)$, E being Young's modulus and ν being Poisson's ratio. Additionally, use of the J path independent integral approach [12] gives the value of the crack tip stress intensity, K_c, associated with the attainment of this state as

$$K_c = \left[E_0 \int_0^{\delta_c} p(u) \mathrm{d}u \right]^{1/2} \tag{2}$$

where u is the relative displacement of the crack faces and $p(u)$ is the restraining stress. With a linear softening law for which

$$p(u) = p_c \left(1 - \frac{u}{\delta_c} \right) \tag{3}$$

Equations 2 and 3 give

$$K_c = \left(\frac{E_0 p_c \delta_c}{2} \right)^{1/2} \tag{4}$$

With such a linear law, Equations 1 and 4 show that the fully developed softening-zone size can be equivalently expressed in the form

$$R_c \sim \frac{0.80 K_c^2}{p_c^2} \tag{5}$$

Carpinteri [7], who assumed a linear softening law, examined the behaviour of a concrete-like material with $E_0 = 400\,000\ \mathrm{kg\,cm^{-2}}$ (~ 400 MPa), $p_c = 40\ \mathrm{kg\,cm^{-2}}$ ($\sim 4 \times 10^{-2}$ MPa) and $\delta_c = 0.005$ cm, whereupon Equation 1 gives $R_c \sim 20$ cm; the fully developed softening-zone size is therefore large, even though the toughness (see Equation 4) associated with a fully developed softening zone is very small ($2 \times 10^{-2}\ \mathrm{MPa\,m^{1/2}}$), this being primarily due to the low p_c value. Foote *et al.* [2] examined the behaviour of a cellulose/asbestos fibre-reinforced mortar with $E_0 = 6000$ MPa, $p_c = 6$ MPa and $\delta_c = 0.08$ cm; assuming that the matrix has no fracture resistance, Equation 1 gives $R_c = 32$ cm, again a very large value, even though the toughness associated with a fully developed softening zone (see Equation 4) is low ($3.79\ \mathrm{MPa\,m^{1/2}}$).

The magnitude of R_c provides a guide as to when the material behaviour can be regarded as being linearly elastic, with crack extension being viewed in terms of the stress intensity factor, K, being equal to K_c, as given by Equation 2 for a general softening law, and by Equation 4 for a linear softening law. The required condition is that R_c should be small in comparison with the characteristic dimensions of the solid under consideration. This condition is satisfied with the materials investigated by Carpinteri [7] and Foote *et al.* [2], only when these characteristic dimensions are very large indeed. However, with some materials, R_c can be fairly small; thus at the other extreme of material behaviour, with a ceramic for which the restraining stress between the crack faces is provided by the untangling of interlocking crystals [1], typical input values are $E_0 = 350 \times 10^3$ MPa, $p_c = 30$ MPa, $\delta_c = 2\ \mu$m, and then Equation 1 gives $R_c \sim 1$ cm, which is much smaller than the R_c values for the materials studied by Carpinteri [7] and Foote *et al.* [2]. The remainder of this paper will be devoted to the situation where the crack extension condition is $K = K_c$ and the material behaviour is linearly elastic; the concern is, therefore, with the case where the softening-zone size is small in comparison with the characteristic dimensions of a solid.

3. General criterion for a cusp catastrophe type of displacement control crack-growth instability

Assume that an elastic solid of thickness B deforms under Mode I plane strain conditions, the cracked solid being subjected to a load, P, which generates a load-point displacement, Δ, that is related to P via the relation

$$\Delta = C_M P \tag{6}$$

where C_M is a compliance function which is dependent on the crack size a; C_M is related to the stress intensity, K, by the standard relation

$$\frac{\mathrm{d}C_M}{\mathrm{d}a} = \frac{2BK^2}{E_0 P^2} = \frac{2H^2}{E_0 B} \tag{7}$$

if K is expressed in the form $K = HP/B$ where H is a function of the crack size. Because crack extension is assumed to occur when $K = K_c$, the crack extension condition is

$$K_c = \frac{HP}{B} \tag{8}$$

Differentiation of Equations 6 and 8 gives respectively

$$\delta\Delta = C_M \delta P + \frac{\mathrm{d}C_M}{\mathrm{d}a} P \delta a \tag{9}$$

and

$$0 = H\delta P + \frac{\mathrm{d}H}{\mathrm{d}a} P \delta a \tag{10}$$

whereupon elimination of δa between Equations 9 and 10 gives

$$\frac{\delta\Delta}{\delta P} = C_M - \left[H \left(\frac{\mathrm{d}C_M}{\mathrm{d}a} \bigg/ \frac{\mathrm{d}H}{\mathrm{d}a} \right) \right] \tag{11}$$

It follows from Equations 7, 10 and 11 that the condition for there to be a cusp catastrophe type of displacement control instability, i.e. both the load, P, and the displacement, Δ, decrease at the onset of crack extension (Fig. 1), is

$$\frac{\mathrm{d}H}{\mathrm{d}a} > 0 \tag{12}$$

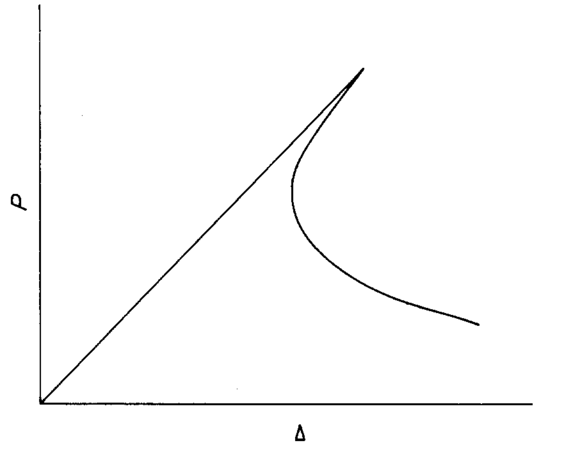

Figure 1 A schematic representation of a cusp catastrophe type of displacement control crack instability at the onset of crack extension.

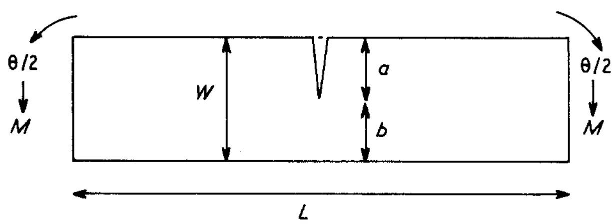

Figure 2 The configuration analysed in the paper; the solid thickness in the direction of the figure normal is B.

and

$$C_M > 2\left(\frac{dC_M}{da}\right)^2 \Big/ \left(\frac{d^2C_M}{da^2}\right) \qquad (13)$$

The preceding general analysis has been with regard to a loading condition where a load P generates a load-point displacement, Δ. However, the same conditions (Equations 12 and 13) apply to the case where an applied moment, M, generates a rotation θ, with $\theta = C_M M$ and $K = HM/B$.

4. The behaviour of an edge-cracked beam subjected to bending deformation

This section analyses the behaviour of a rectangular beam of length, L, thickness, B, and width, W, containing an edge crack with depth a at the beam midsection (Fig. 2). The ends of the beam are subjected to a relative rotation θ, which is associated with a moment M. The stress intensity, K, for this configuration, assuming pure bending, is given in the form [13]

$$K = \frac{6M(\pi a)^{1/2}}{BW^2} F\left(\frac{a}{W}\right) \qquad (14)$$

where $F(a/W)$ has been expressed in graphical form [13]. Consequently, the crack extension condition $K = K_C$ becomes

$$K_C = \frac{6M(\pi a)^{1/2}}{BW^2} F\left(\frac{a}{W}\right) \qquad (15)$$

TABLE I Values of M and θ for a range of a/W values and for two specific L/W values

$\frac{a}{W}$	$F\left(\frac{a}{W}\right)$	$\frac{M}{BW^{3/2}K_c}$	$S\left(\frac{a}{W}\right)$	$\frac{E_0 W^{1/2}\theta}{12K_c}$	
				$L/W = 4$	$L/W = 10$
0.1	1.042	0.285	0.062	0.073	0.029
0.2	1.048	0.201	0.234	0.056	0.021
0.3	1.109	0.155	0.510	0.049	0.017
0.4	1.247	0.119	0.972	0.044	0.014
0.5	1.497	0.088	1.800	0.042	0.012
0.6	1.917	0.063	3.250	0.041	0.010
0.7	2.774	0.041	6.667	0.044	0.010
0.8	4.831	0.022	16.000	0.049	0.009
0.9	12.500	0.008	66.000	0.068	0.011

Furthermore, the relative rotation, θ, of the ends of the beam is given by the expression

$$\theta = \frac{ML}{E_0 I} + \frac{24M}{E_0 BW^2} S\left(\frac{a}{W}\right) \qquad (16)$$

where the first term on the right-hand side is the contribution from the uncracked configuration, with $I \equiv BW^3/12$ being the beam's moment of inertia. The second term is due to the crack, with the function $S(a/W)$ being expressed in graphical form [13]. Equations 15 and 16 can be re-written in the forms

$$\frac{M}{BW^{3/2}K_C} = \frac{1}{6\pi^{1/2}} \cdot \left[1 \Big/ \left(\frac{a}{W}\right)^{1/2} F\left(\frac{a}{W}\right)\right] \qquad (17)$$

$$\frac{E_0 W^{1/2}\theta}{12K_C} = \left[\frac{L}{W} + 2S\left(\frac{a}{W}\right)\right] \Big/ 6\pi^{1/2}\left(\frac{a}{W}\right)^{1/2} F\left(\frac{a}{W}\right) \qquad (18)$$

These relations allow both M and θ to be obtained for the complete spectrum of (a/W) values; the results are given in Table I for the two cases $L/W = 4$ (the case analysed by Carpinteri [7]) and $L/W = 10$. The results are also shown in Fig. 3, the arrowed points referring to specific a/W values. Recognizing that M and θ both increase and are linearly related during loading prior to crack extension, Fig. 3 shows that there is a cusp catastrophe type of displacement control crack-growth instability at the onset of crack extension if $a/W \gtrsim 0.6$ when $L/W = 4$ and if $a/W \gtrsim 0.8$ when $L/W = 10$. The range of a/W values over which there is a cusp instability is, therefore, broader as the ratio L/W increases. The analysis shows that this range is a function only of the geometrical configuration (see Equation 18) and does not depend on the material's properties (K_c). Furthermore, the criterion for a cusp instability depends on the two dimensionless parameters, a/W and L/W, but not on the dimensions themselves.

To supplement these results, it is instructive to concentrate on the special case where the remaining ligament width $b = W - a$ (see Fig. 2) is very small. In this case, the stress intensity, K, is given in the form [13]

$$K = \frac{3.975M}{Bb^{3/2}} \qquad (19)$$

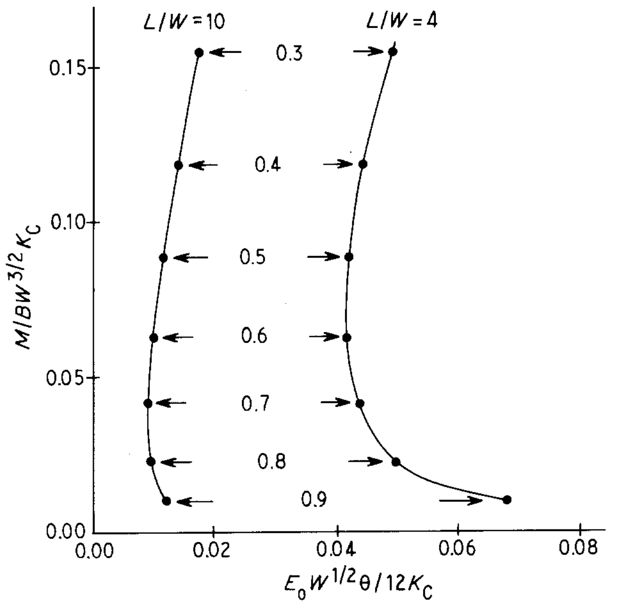

Figure 3 The relation between the moment, M, and relative rotation, θ, of the ends of the beam during crack extension, for two specific L/W values; the arrowed points refer to specific a/W values.

whereupon the crack extension condition becomes

$$K_C = \frac{3.975M}{Bb^{3/2}} \tag{20}$$

Furthermore the relative rotation, θ, of the ends of the beam is given by the expression [13]

$$\theta = \frac{ML}{E_0 I} + \frac{15.8M}{E_0 Bb^2} \tag{21}$$

where the first term on the right-hand side is the contribution from the uncracked configuration, with $I \equiv BW^3/12$ again being the beam's moment of inertia; the second term is due to the crack. To determine the condition for a cusp catastrophe type of displacement control crack-growth instability, we can proceed as in this section's earlier analysis for a general a/W. However instead, but equivalently, we will analyse the problem as a special case of the preceding section's general analysis. Thus comparing Equation 21 with Equation 6, when it is applied to the moment-rotation situation, i.e. Δ and P are replaced by, respectively, θ, and M, it follows that

$$C_M = \frac{L}{E_0 I} + \frac{15.8}{E_0 Bb^2} \tag{22}$$

and then, noting that $\delta a = -\delta b$, Equation 13 shows that the criterion for a cusp catastrophe type of displacement control crack-growth instability is

$$\frac{b}{W} \gtrsim 0.66 \Big/ \left(\frac{L}{W}\right)^{1/2} \tag{23}$$

This relation shows that the critical b/W ratio above which there is a cusp-type instability decreases as the ratio L/W increases, a conclusion that is consistent with that reached via the earlier analysis leading to Fig. 3. Indeed for both $L/W = 4$, and $L/W = 10$, the results given by Equation 23 are consistent with the curves in Fig. 3.

5. Discussion

This work was concerned with the condition for a cusp catastrophe type of displacement control crack-growth instability in an elastic-softening material, with regard to an edge-cracked beam that was subjected to bending deformation by an applied relative rotation of the ends of the beam. The work focused on the behaviour of a material whose fully developed softening-zone size is very small when compared with the solid's characteristic dimensions, thus complementing the work of Carpinteri [7], who analysed the behaviour of a material whose fully developed softening size is large (he also considered three-point bending deformation, in contrast to the present paper's uniform bending deformation, but this difference is not important). The present paper's theoretical results clearly show that the criterion for a cusp-type instability is independent of the material properties, i.e. K_c, but depends solely on the configuration's length parameters. The criterion can be expressed in the form

$$\frac{a}{W} < g\left(\frac{L}{W}\right) \tag{24}$$

where a is the crack depth, W the beam width and L the beam length, with $g(L/W)$ being an increasing function of L/W; the criterion is $a/W \lesssim 0.6$ for $L/W = 4$ and $a/W \lesssim 0.8$ for $L/W = 10$. It is immediately observed that the criterion for a cusp-type instability depends on the ratios a/W and L/W but not on the actual magnitudes of the dimensions. The criterion therefore remains the same if the solid dimensions are scaled proportionally, a result that is in contrast to the results [7] for a material with a large softening zone size, for which the criterion for a cusp-type instability is not as simple as Equation 17. Although the present analysis has been restricted to the edge-cracked bend specimen configuration, similar conclusions are expected for other configurations where the crack tip stress intensity increases with crack size for a fixed loading, i.e. Equation 12. Namely, that provided the softening-zone size is very small, the criterion for a cusp catastrophe type of displacement control crack-growth instability is independent of the material properties, but is dependent on the ratios of the configuration's geometrical parameters though not on the actual magnitudes of these parameters. Furthermore the range of crack sizes over which there is a cusp-type instability increases with the configuration's compliance (i.e. as L/W increases with the present paper's configuration).

It is important to emphasize that this work has been concerned with the behaviour of a material whose softening-zone size was very small, whereas Carpinteri's study [7] focused on the behaviour of a material whose softening-zone size was large; extensions to this work will attempt to bridge these two extremes of material behaviour. Thus Part II [14] analyses the model of an infinite solid containing two symmetrically situated deep cracks and with tensile loading of the small remaining ligament. With this model, a simple analytical treatment is possible across the complete spectrum of material behaviour, provided that the stress within the softening zone retains a constant value of p_c. It is, therefore, possible to ascertain how the criterion for a cusp-type instability depends, in a coupled manner, on both material and geometrical parameters. It is also the intention to develop a type of "small-scale yielding" analysis which is applicable to a material whose softening zone is not infinitesimally small, as has been assumed in the present analysis, but instead is a small fraction of the solid's characteristic dimensions; such a general analysis should be applicable to any configuration.

Finally, it is worth emphasizing that the importance of a cusp-type instability stems from the fact that many engineering structures are subjected to displacement control loading. With such an instability, the load drops immediately, and although there is stability on the lower portion of the load–displacement curve see (Fig. 1) according to the results of a static analysis, the energy associated with the load reduction may well lead to catastrophic dynamic failure of the structure. This possibility is the motivation for research in this particular area of materials engineering.

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